Stability of solutions of chemotaxis equations in reinforced random walks

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Abstract

In this paper we consider a nonlinear system of differential equations consisting of one parabolic equation and one ordinary differential equation. The system arises in chemotaxis, a process whereby living organisms respond to chemical substance by moving toward higher, or lower, concentrations of the chemical substance, or by aggregating or dispersing. We prove that stationary solutions of the system are asymptotically stable.

Keywords: Chemotaxis; Reinforced random walk; Parabolic equations; Stability of stationary solutions

1. Introduction

Chemotaxis is the phenomenon whereby living organisms respond to chemical substance by motion and rearrangement (*taxis*). They may move toward the higher concentration of the chemical substance (positive taxis), or away from it (negative taxis), they may aggregate, or they may disperse.

A model that leads to aggregation of certain types of bacteria has been set up by Keller and Segel [11,12]. The model involves the density distribution p of the bacteria and the chemical concentration w in a coupled system of partial differential equation,

$$\frac{\partial p}{\partial t} = \Delta p - \operatorname{div}\big(p\chi(w)\nabla w\big),$$
$$0 = \Delta w + (p - 1).$$

This system was studied in [1,2,6–10,18,20] (see also [22]).

Another model, called reinforced random walk (after Davis [4]), was more recently developed by Othmer and Stevens [19]. The motivation of this model was to gain understanding of the mechanism that causes the aggregation of myxobacteria. These common soil bacteria slide over slime trails thereby reinforcing the trails. Working first with a discrete number of steps, the model stipulates that the decision of the walker with conditional probability $p_n(t)$, at the nth site at time t, as to when and where to jump is affected by the densities of the control species, $w_m(t)$. As the size of the random steps shrinks to zero, Othmer and Stevens derive a system of equations

$$\frac{\partial p}{\partial t} = \operatorname{div}\big(D\nabla p - p\chi(w)\nabla w\big), \tag{1.1}$$
$$\frac{\partial w}{\partial t} = g(p, w), \tag{1.2}$$

where D is the diffusion constant and $\chi(w)$ is the chemotactic sensitivity of the bacteria. Both $\chi(w)$ and $g(p, w)$ depend on the nature of the interaction between the bacteria and the chemical stimulus. In a very recent paper, Stevens [21] introduced a general stochastic many-particle system and rigorously derived chemotactic equations of the form

$$\begin{aligned} \frac{\partial p}{\partial t} &= \operatorname{div}\big(\mu\nabla p - \chi(p, w)p\nabla w\big), \\ \frac{\partial w}{\partial t} &= \varepsilon\Delta w + \beta(p, w)p - \gamma(p, w)w, \end{aligned} \tag{1.3}$$

with $\varepsilon > 0$.

A chemotaxis process occurs also in the growth of a tumor. The tumor secretes chemical species that attract the nearby endothelial cells, which form the surface of capillary blood vessels. In this way new blood vessels sprout towards the tumor and begin to provide it with additional nourishment. The phenomenon of sprouting of new blood vessels is called angiogenesis.

Recently, Levine et al. [14,16] developed models of angiogenesis based on analysis of the relevant biochemical processes and on the methodology of the reinforced random walk of [19]. Their model involves several diffusing populations and several chemical species. Another model of angiogenesis with one diffusing population and two nondiffusing ones was studied by Anderson and Chaplain [3].

In this paper we consider the system (1.1), (1.2) for general functions $\chi(w)$, $g(p, w)$. We prove that

$$\text{any stationary solution } (p_*, w_*) \text{ is asymptotically stable,} \tag{1.4}$$

provided

$$g = \varphi h, \quad \varphi > 0, \quad h_p > 0, \quad p\chi h_p + h_w < 0 \quad \text{at } (p_*, w_*). \tag{1.5}$$

More precisely, if (1.5) holds then any solution of (1.1), (1.2) in a bounded domain Ω, with boundary condition

$$D\frac{\partial p}{\partial n} - p\chi\frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega$$

and with initial values near (p_*, w_*), exists for all $t > 0$ and converges, as $t \to \infty$, to a nearby stationary solution $(\bar{p}, \bar{w})$. For simplicity we shall always take $D = 1$.

The assertion (1.4) means that, under the assumption (1.5), chemotaxis leads to *uniform* distribution as $t \to \infty$ provided the initial distribution is nearly uniform. We shall also prove a similar result for more general initial distributions (under stronger assumptions than (1.5)).

The proof of (1.4) consists of three steps. In the first step (Section 3) we establish a priori bounds; in the second step (Section 4) we prove the existence and uniqueness of a global solution; and in the third step (Section 5) we prove that any solution with initial data near (p_*, w_*) converges to a stationary solution as $t \to \infty$.

Section 6 extends some of these results to the case where there are several chemical species, and also to some chemotaxis equations of the form (1.3).

In Section 7 we give several examples from among those that appear in [5,15, 19]. We also give an application related to the angiogenesis model of [3].

2. The main results

Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{2+\beta}$ boundary $\partial\Omega$, $0 < \beta < 1$. Consider the differential system

$$\frac{\partial p}{\partial t} = \operatorname{div}\big(\nabla p - p\chi(w)\nabla w\big), \quad x \in \Omega,\ t > 0, \tag{2.1}$$

$$\frac{\partial w}{\partial t} = g(p, w), \quad x \in \Omega,\ t > 0, \tag{2.2}$$

with the boundary conditions

$$\frac{\partial p}{\partial n} - p\chi(w)\frac{\partial w}{\partial n} = 0, \quad x \in \partial\Omega,\ t > 0, \tag{2.3}$$

where $\partial p/\partial n$ is the outward normal derivative, and initial conditions

$$p(x, 0) = p_0(x), \quad w(x, 0) = w_0(x), \quad x \in \Omega. \tag{2.4}$$

We assume that

$$p_0(x) > 0, \quad \text{for } x \in \overline{\Omega}.$$

The function $\chi(w)$ is the chemotactic sensitivity function of the organisms. We first consider the case

$$\chi(w) > 0 \quad \text{for } -\infty < w < \infty. \tag{2.5}$$

For $w > 0$ this condition means that the organisms react positively toward higher concentration of the chemical substance. The case $w < 0$ is not immediately relevant here, but it is convenient to include it in order to deal later on with the case of negative chemotactic sensitivity functions.

The function $g(p, w)$ is assumed to have the form

$$g(p, w) = \varphi(p, w)h(p, w), \tag{2.6}$$

where, for some constants

$$0 \leqslant p_1 < p_2, \qquad w_1 < w_2,$$

there holds

$$\varphi(p, w) > 0 \quad \text{if } p_1 \leqslant p \leqslant p_2,\ w_1 \leqslant w \leqslant w_2, \tag{2.7}$$

$$h(p_1, w_1) = h(p_2, w_2) = 0. \tag{2.8}$$

Note that (p_i, w_i) is a stationary solution of (2.1)–(2.3). Note also that w_1, w_2 can be any real numbers. We shall further assume that

$$\varphi, \chi, h \text{ are in } C^1 \quad \text{for } 0 \leqslant p < \infty,\ -\infty < w < \infty, \tag{2.9}$$

$$\frac{\partial h}{\partial p} > 0 \quad \text{if } p_1 \leqslant p \leqslant p_2,\ w_1 \leqslant w \leqslant w_2, \tag{2.10}$$

$$p\chi \frac{\partial h}{\partial p} + \frac{\partial h}{\partial w} < 0 \quad \text{if } p_1 \leqslant p \leqslant p_2,\ w_1 \leqslant w \leqslant w_2. \tag{2.11}$$

Introducing the function

$$f(w) = \exp\left[\int_{w_1}^{w} \chi(s)\, ds\right], \tag{2.12}$$

we define a new variable q by

$$p = f(w)q, \tag{2.13}$$

and set

$$q_i = \frac{p_i}{f(w_i)} \quad (i = 1, 2). \tag{2.14}$$

We claim that

$$q_1 < q_2. \tag{2.15}$$

Indeed, by (2.10) and (2.11) we have $h_p > 0$, $h_w < 0$. We can therefore solve the equation $h(p,w) = 0$ in the form $p = \Psi(w)$, where $\Psi'(w) = -h_w/h_p > 0$. But then

$$q_2 - q_1 = \frac{\Psi(w_2)}{f(w_2)} - \frac{\Psi(w_1)}{f(w_1)} > 0, \tag{2.16}$$

since

$$\begin{aligned}\left(\frac{\Psi(w)}{f(w)}\right)' &= \frac{1}{f(w)}\left(\Psi'(w) - \Psi(w)\frac{f'(w)}{f(w)}\right) \\ &= \frac{1}{f(w)}\left(-\frac{h_w}{h_p} - p\chi(w)\right) > 0 \end{aligned}\tag{2.17}$$

by (2.10), (2.11).

In terms of the variables q, w, the system (2.1)–(2.3) becomes

$$\begin{aligned}\mathcal{L}q &\equiv \frac{\partial q}{\partial t} - \Delta q - \chi(w)\nabla w \cdot \nabla q \\ &= -q\chi(w)\varphi\big(qf(w), w\big)h\big(qf(w), w\big), \quad x \in \Omega,\ t > 0,\end{aligned}\tag{2.18}$$

$$\frac{\partial w}{\partial t} = g\big(qf(w), w\big), \quad x \in \Omega,\ t > 0, \tag{2.19}$$

and

$$\frac{\partial q}{\partial n} = 0, \quad x \in \partial\Omega,\ t > 0. \tag{2.20}$$

The initial conditions (2.4) become

$$q(x,0) = q_0(x), \qquad w(0,x) = w_0(x), \tag{2.21}$$

where $q(x,0) = p_0(x)/f(w_0(x))$.

For simplicity we assume that

$$\begin{aligned}&p_0(x), w_0(x) \text{ belong to } C^{2+\beta}(\Omega) \quad (0 < \beta < 1), \quad \text{and} \\ &\frac{\partial p_0}{\partial n} - p_0\chi(w_0)\frac{\partial w_0}{\partial n} = 0 \quad \text{on } \partial\Omega.\end{aligned}\tag{2.22}$$

The additional and more crucial assumption on the initial data is that

$$q_1 < q_0(x) < q_2, \qquad w_1 < w_0(x) < w_2. \tag{2.23}$$

Set

$$\Omega_T = \Omega \times (0,T) \quad (0 < T \leqslant \infty).$$

In Section 3 we prove the following a priori bounds.

Theorem 2.1. *Under the assumptions* (2.5)–(2.11) *and* (2.22), (2.23), *if* (p,w) *is a solution of* (2.1)–(2.4) *in* Ω_T, *then*

$$q_1 \leqslant q(x,t) \leqslant q_2, \quad w_1 \leqslant w(x,t) \leqslant w_2 \quad \text{in } \Omega_T. \tag{2.24}$$

The inequalities in (2.24) imply that $p_1 \leqslant p \leqslant p_2$ in Ω_T.

In Section 4 we shall use Theorem 2.1 to prove the following existence theorem.

Theorem 2.2. *Under the assumptions* (2.5)–(2.11) *and* (2.22), (2.23), *there exists a unique global solution* (p, w) *of* (2.1)–(2.4) *with*

$$p, w \text{ in } C_{x,t}^{2+\beta,1+\beta/2}(\Omega_\infty). \tag{2.25}$$

Integrating (2.1) over Ω and using (2.3) we get

$$\int_\Omega p(x,t)\,dx = \int_\Omega p_0(x)\,dx, \quad t > 0. \tag{2.26}$$

We introduce the quantity

$$\bar{p} = \frac{1}{|\Omega|}\int_\Omega p_0(x)\,dx, \tag{2.27}$$

where $|\Omega| =$ volume of Ω.

Now let (p_*, w_*) be a stationary solution (a priori not necessarily constant) with

$$p_1 < p_* < p_2, \qquad w_1 < w_* < w_2. \tag{2.28}$$

Then $h(p_*, w_*) = 0$ and, since $h_p > 0$, $h_w < 0$, we can write $p_* = \Psi(w_*)$, where

$$\frac{d\Psi}{dw} = -\frac{h_w}{h_p} \quad \text{at } (p_*, w_*).$$

Substituting this into (1.1) we get

$$\operatorname{div}\left(\frac{1}{h_{p_*}}\big(h_{w_*} + p_*\chi(w_*)h_{p_*}\big)\nabla w_*\right) = 0 \quad \text{in } \Omega.$$

In view of (2.10), (2.11), this equation is elliptic. Since also $\partial w_*/\partial n = 0$ on $\partial\Omega$, it follows that $w_* =$ constant, and then also $p_* =$ constant. Thus any stationary solution satisfying (2.24) is constant.

Introducing the number

$$q_* = \frac{p_*}{f(w_*)},$$

we consider initial values “near” the stationary solution, in the sense that

$$\big|p_0(x) - p_*\big| \leqslant \varepsilon, \qquad \big|w_0(x) - w_*\big| \leqslant \varepsilon, \tag{2.29}$$

where ε is sufficiently small. Then

$$\big|q_0(x) - q_*\big| \leqslant C\varepsilon \tag{2.30}$$

for some constant C. Since $\partial h/\partial p > 0$ and $\partial h/\partial w < 0$, there exists a unique solution $\bar{w}$ to the equation

$$h(\bar{p}, \bar{w}) = 0 \quad (w_1 < \bar{w} < w_2), \tag{2.31}$$

where $\bar{p}$ is defined in (2.27).

In Section 5 we prove the following asymptotic stability result for stationary solutions.

Theorem 2.3. *If* (2.29) *holds with* ε *sufficiently small then the solution* $p(x,t)$, $w(x,t)$ (*established in Theorem* 2.2) *has the following asymptotic behavior*:

$$\int_\Omega |p - \bar{p}|^2 \, dx \to 0, \qquad \int_\Omega \left|w(x,t) - \bar{w}\right|^2 dx \to 0 \quad \text{as } t \to \infty. \tag{2.32}$$

Remark 2.1. The proof of Theorem 2.3 requires the conditions (2.10), (2.11) only for $p_2 - p_1$ and $w_2 - w_1$ small, where $w_1 < w_* < w_2$, $p_1 < p_* < p_2$. Thus, we may actually replace (2.10), (2.11) by the inequalities

$$h_p > 0, \quad p\chi h_p + h_w < 0 \quad \text{at the point } (p_*, w_*). \tag{2.33}$$

As will be seen, the proof of Theorem 2.3 yields the following more global asymptotic stability result in case $\varphi \equiv 1$.

Theorem 2.4. *Let the assumptions of Theorem* 2.1 *hold and assume also that* $\varphi \equiv 1$. *Then the solution* (*established in Theorem* 2.2) *has the asymptotic behavior* (2.32).

Note that in this theorem there is no smallness restriction on the size of the quantities $w_2 - w_1$, $p_2 - p_1$, but (2.23) must be satisfied.

Consider next the case of negative taxis, that is,

$$\chi(w) < 0, \tag{2.34}$$

and replace (2.10) by

$$\frac{\partial h}{\partial p} < 0, \quad \text{if } p_1 \leqslant p \leqslant p_2, \ w_2 \leqslant w \leqslant w_1. \tag{2.35}$$

Theorem 2.5. *Theorems* 2.1–2.4 *remain valid in case* (2.5) *and* (2.10) *are replaced by* (2.34) *and* (2.35), *and the roles of* w_1, w_2 *are interchanged in all the other assumptions and assertions.*

Indeed, setting

$$\tilde{w} = -w, \qquad \tilde{\chi}(\tilde{w}) = -\chi(w), \qquad \tilde{g}(p, \tilde{w}) = -g(p, w),$$
$$\tilde{\varphi}(p, \tilde{w}) = \varphi(p, w), \qquad \tilde{h}(p, \tilde{w}) = -h(p, w),$$

Eqs. (2.1), (2.2) reduce to

$$\frac{\partial p}{\partial t} = \operatorname{div}\big(\nabla p - p\tilde{\chi}(\tilde{w})\nabla\tilde{w}\big),$$
$$\frac{\partial \tilde{w}}{\partial t} = \tilde{g}(p, \tilde{w}),$$

where $\tilde{g}(p, \tilde{w}) = \tilde{\varphi}(p, \tilde{w})\tilde{h}(p, \tilde{w})$, $\tilde{\varphi} > 0$, and $\tilde{\chi}, \tilde{h}$ satisfy the assumptions of Theorems 2.1–2.3.

3. Proof of Theorem 2.1

We first prove the inequalities

$$q(x,t) \leqslant q_2, \qquad w(x,t) \leqslant w_2. \tag{3.1}$$

Denote the right-hand side of (2.18) by $\Psi(q, w)$. For any small $\delta > 0$, consider the system

$$\mathcal{L}q \equiv \Psi(q, w) - \delta \quad \text{in } \Omega_T, \tag{3.2}$$
$$\frac{\partial w}{\partial t} = g\big(qf(w), w\big) - \delta \quad \text{in } \Omega_T, \tag{3.3}$$

together with (2.20), (2.21), and denote its solution by (q_δ, w_δ). The existence of this solution follows by considering the pair $Q = q_\delta - q$, $W = w_\delta - w$ which satisfies a perturbed system about (q, w). We may view (Q, W) as a solution to a fixed point transformation which, for δ small, is a contraction. Since the proof of this fact is quite standard, we omit the details. The proof also shows that $q_\delta \to q$ and $w_\delta \to w$ pointwise as $\delta \to 0$. We claim that

$$q_\delta(x,t) < q_2 \quad \text{and} \quad w_\delta(x,t) < w_2 \tag{3.4}$$

in Ω_T. Suppose this is not true. Then there is a point (x_0, t_0) such that (3.4) holds in Ω_{t_0} for some $0 < t_0 \leqslant T$, and either

$$q_\delta(x_0, t_0) = q_2, \tag{3.5}$$

or

$$w_\delta(x_0, t_0) = w_2. \tag{3.6}$$

Consider first the case (3.5). By the maximum principle and (2.20), $x_0 \notin \partial\Omega$ and, by the mean value theorem,

$$h\big(q_2 f(w_\delta), w_\delta\big) = h\big(q_2 f(w_2), w_2\big) + \left[\frac{dh}{dw}\big(q_2 f(w), w\big)\right]_{w=\tilde{w}} (w_\delta - w_2)$$
$$\text{at } (x_0, t_0),$$

where $w_\delta \leqslant \tilde{w} \leqslant w_2$. Recalling that $p_2 = q_2 f(w_2)$ and $h(p_2, w_2) = 0$ and setting $\tilde{p} = q_2 f(\tilde{w})$, we get

$$h\bigl(q_2 f(w_\delta), w_\delta\bigr) = (w_\delta - w_2)\bigl[\tilde{p}\chi(\tilde{w})h_p(\tilde{p}, \tilde{w}) + h_w(\tilde{p}, \tilde{w})\bigr] \geqslant 0, \tag{3.7}$$

by (2.11). Since q_δ takes its maximum in $\overline{\Omega}_{t_0}$ at (x_0, t_0), we have $\mathcal{L}q_\delta \geqslant 0$ at (x_0, t_0). On the other hand, by (3.2), the definition of $\Psi(q, w)$, (3.7) and the inequality $\chi(w_\delta) > 0$ we have $\mathcal{L}q_\delta < 0$ at (x_0, t_0), which is a contradiction. We conclude that $q_\delta(x_0, t_0) < q_2$.

Consider next the case (3.6). Then $(\partial/\partial t)w_\delta(x_0, t_0) \geqslant 0$, so that

$$h\bigl(q_\delta f(w_2), w_2\bigr) > 0,$$

by (3.3). But since $q_\delta < q_2$ and $\partial h/\partial p > 0$, we get

$$h\bigl(q_2 f(w_2), w_2\bigr) > 0$$

which contradicts (2.8).

Having proved (3.4) we now let $\delta \to 0$ and obtain the inequalities $q \leqslant q_2$, $w \leqslant w_2$, in Ω_T. The proof of $q \geqslant q_1$, $w \geqslant w_1$ is similar, replacing $-\delta$ by $+\delta$ in (3.2), (3.3). □

4. Proof of Theorem 2.2

Denote the right-hand side of (2.18) by $\Phi(p, w)$. Given $(\tilde{p}, \tilde{w})$ we solve

$$\mathcal{L}q = \Phi(\tilde{p}, \tilde{w}), \qquad w_t = g(\tilde{p}, \tilde{w})$$

in Ω_T with the boundary and initial conditions (2.20), (2.21), and define a mapping S by

$$S(\tilde{p}, \tilde{w}) = (p, w) \quad \text{where } p = f(w)q.$$

Using the Schauder estimates [13] one can prove that if T is sufficiently small then S is a contraction and thus it has a unique fixed point in the Hölder class

$$C_{x,t}^{1+\beta,(1+\beta)/2}(\Omega_T),$$

and that the solution actually belongs to

$$C_{x,t}^{2+\beta,1+\beta/2}(\Omega_T). \tag{4.1}$$

Since the proof is standard (see, for instance, [5]) we omit the details. The proof also shows that T depends only on the $(C_x^{2+\beta}(\Omega_T)$ norm of the initial data. Hence, given any T_0 and $0 < T < T_0$, if we can establish the a priori bounds

$$|p|_{C_{x,t}^{2+\beta,1+\beta/2}(\Omega_T)} \leqslant C, \qquad |w|_{C_{x,t}^{2+\beta,1+\beta/2}(\Omega_T)} \leqslant C \tag{4.2}$$

for the solution (assuming it exists in Ω_T) with a constant C which is independent of T, then we can extend the solution step-by-step to Ω_{T_0}. Since T_0 is arbitrary, this will establish the existence of a global solution, which is clearly unique (by the fixed point argument).

In order to prove (4.2) we write Eq. (2.18) in a “nearly” divergence form

$$bq_t - \operatorname{div}(f\nabla q) = f\Phi, \tag{4.3}$$

where $b = f$ and $\Phi = \Phi(p, w)$ is as defined above. By Theorem 2.1

$$b \geqslant c_0, \qquad |b_t| \leqslant C_0, \qquad |\Phi| \leqslant C_0, \tag{4.4}$$

where c_0 and C_0 are positives constants independent of T and T_0. In case $b = 1$, Theorem 10.1 in [13, p. 204] yields the estimate

$$|q|_{C^{\alpha,\alpha/2}_{x,t}(\Omega_T)} \leqslant C \quad \text{for some } \alpha > 0, \tag{4.5}$$

where C is a constant independent of T. The proof in [13] is actually given for a solution with zero boundary values on $\partial\Omega \times (0, T)$, but the same proof is valid in the case (2.20) of zero normal derivatives.

If $b \not\equiv 1$ but satisfies the inequalities in (4.4), the proof given in [13, p. 204] needs to be slightly modified. The additional integral that we now get, after performing integration by parts on the integral

$$\iint_{\Omega_T} (q-k)^+ bq_t \, dx\, dt \quad (k \text{ is a real number})$$

which occurs in that proof, is

$$-\iint_{\Omega_T} (q-k)^+ b_t q \, dx\, dt$$

and this is majored by other expressions. Using (4.5) we can next estimate the C^α norm of w from (2.19) and, in fact, conclude that

$$|w|_{C^{\alpha,\alpha/2}_{x,t}(\Omega_T)} + |w_t|_{C^{\alpha,\alpha/2}_{x,t}(\Omega_T)} \leqslant C. \tag{4.6}$$

We next wish to prove that

$$|D_x q|_{C^{\alpha,\alpha/2}_{x,t}(\Omega_T)} \leqslant C, \tag{4.7}$$

where C is again a constant independent of T.

For the case where $b = 1$ in (4.3), this follows from Theorem 4.21 in [17, p. 69]. We shall briefly indicate how the proof can be extended to the case $b = f(w)$, provided w satisfies (4.6). As in [17] we first want to estimate the interior $C^{1+\alpha,\alpha/2}_{x,t}(\Omega_T)$ norm of q, which we shall denote by $|q|_{1+\alpha}$, in terms of the C^1 norm $|q|_1$. That is, we want to prove, as in [17, p. 57], that

$$|q|_{1+\alpha} \leqslant C|q|_1 + C. \tag{4.8}$$

The proof of (4.8) for the case $b = 1$ is based on deriving integral estimates for functions v (which are denoted there by w) vanishing on the parabolic boundary of domains

$$\Omega_{t_1,t_2} \equiv \Omega_0 \times \{t_1 < t < t_2\}$$

with small diameter. Use is made of the equation

$$\iint\limits_{\Omega_{t_1,t_2}} b v_t v \, dx \, dt = \frac{1}{2} \int\limits_{\Omega_0} b v^2 \, dx \Big|_{t_1}^{t_2} - \frac{1}{2} \iint\limits_{\Omega_{t_1,t_2}} b_t v^2 \, dx \, dt \tag{4.9}$$

for the case $b \equiv 1$ (see [17, p. 57]). When $b = f(w)$, the first term on the right-hand side of (4.9) is treated in the same way as in case $b = 1$, and the second term is "harmless" and, in fact, in view of (4.6), can be absorbed by

$$\iint\limits_{\Omega_{t_1,t_2}} |Dq|^2 \, dx \, dt.$$

The rest of the proof of (4.8) then proceeds as in the case $b \equiv 1$.

Next we wish to extend (4.8) to a neighborhood of the boundary $\partial\Omega \times (0, T)$. As in [17, p. 77] we flatten the boundary locally by a $C^{2+\beta}$ transformation. The resulting equation for q is

$$b q_t - \partial_i \left(A^{i,j} \partial_j q\right) = f \Phi,$$

where $A^{i,j} \in C^{\alpha,\alpha/2}_{x,t}$. We then extend the proof of the Theorem 4.15 [17, p. 64] from the case $b \equiv 1$ to the case $b = f(w)$ as before.

Combining the interior and boundary estimates, we conclude that

$$|D_x q|_{C^{\alpha,\alpha/2}_{x,t}(\Omega_T)} \leqslant C |D_x q|_{L^\infty(\Omega_T)} + C. \tag{4.10}$$

By the mean value theorem, if $|x - x_0| \leqslant \varepsilon$ then

$$q(x,t) - q(x_0,t) = (x - x_0)\nabla_x q(x_0,t) + O(\varepsilon^{1+\alpha}) |D_x q|_{C^{\alpha,\alpha/2}_{x,t}(\Omega_T)},$$

so that

$$\left|\nabla_x q(x_0,t)\right| < \varepsilon^\alpha |D_x q|_{C^{\alpha,\alpha/2}_{x,t}(\Omega_T)} + \frac{C}{\varepsilon}. \tag{4.11}$$

We now use partition of unity $\{\zeta_j\}$ of $\overline{\Omega}$, and apply (4.10) to $\zeta_j q$, noting that $\zeta_j q$ satisfies the same differential equations as q but with right-hand side bounded by the right-hand side of (4.10). Taking the diameter of $\operatorname{supp} \zeta_j$ to be smaller than ε, and applying also (4.11), we obtain, after summing over j, the bound

$$|D_x q|_{C^{\alpha,\alpha/2}_{x,t}(\Omega_T)} \leqslant C, \tag{4.12}$$

provided ε is chosen small enough.

From (2.19) we easily find that the estimate (4.12) holds also for $D_x w$ and for $D_t D_x w$. We can then apply the Schauder estimates to (2.18) and, together with (2.19), boost the estimates of both q and w to obtain the bounds in (4.2). $\square$

5. Proof of Theorem 2.3

Multiplying (1.1) by p and integrating over Ω_T, we get

$$\frac{1}{2}\int_\Omega p^2\Big|_0^T dx + \iint_{\Omega_T} |\nabla p|^2\, dx\, dt = \iint_{\Omega_T} p\chi(w)\nabla p\cdot\nabla w\, dx\, dt. \tag{5.1}$$

From (2.2) we have

$$\nabla w_t = g_p\nabla p + g_w\nabla w.$$

Taking the scalar product with $\lambda\nabla w$, where λ is a positive number, and integrating over Ω_T, we find that

$$\frac{\lambda}{2}\int_\Omega |\nabla w|^2\Big|_0^T dx = \lambda\iint_{\Omega_T} g_w|\nabla w|^2\, dx\, dt + \lambda\iint_{\Omega_T} g_p\nabla p\cdot\nabla w\, dx\, dt. \tag{5.2}$$

Adding (5.1) to (5.2) results in the relation

$$\begin{aligned}&\frac{\lambda}{2}\int_\Omega \big|\nabla w(x,T)\big|^2 dx + \iint_{\Omega_T} |\nabla p|^2\, dx\, dt + \iint_{\Omega_T} (-\lambda g_w)|\nabla w|^2\, dx\, dt\\ &\quad = \iint_{\Omega_T} \big(p\chi(w) + \lambda g_p\big)\nabla p\cdot\nabla w\, dx\, dt + O(1),\end{aligned} \tag{5.3}$$

where $|O(1)| \leqslant C$, C independent of T. By Schwarz's inequality, the integral on the right-hand side is bounded by

$$(1-\delta)\iint_{\Omega_T} |\nabla p|^2\, dx\, dt + \frac{1}{4(1-\delta)}\iint_{\Omega_T} \big(p\chi(w) + \lambda g_p\big)^2|\nabla w|^2\, dx\, dt$$

for any $0 < \delta < 1$. If we can show that

$$\big(p\chi(w) + \lambda g_p\big)^2 < 4(-\lambda g_w) \tag{5.4}$$

uniformly for (p, w) then, by choosing δ sufficiently small, we conclude from (5.3) the bound

$$\int_\Omega \big|\nabla w(x,t)\big|^2 dx + \iint_{\Omega_\infty} |\nabla p|^2\, dx\, dt + \iint_{\Omega_\infty} |\nabla w|^2\, dx\, dt \leqslant C \tag{5.5}$$

for all $t > 0$.

Consider the quadratic equation in λ,

$$\big(\lambda g_p + p\chi(w)\big)^2 + 4\lambda g_w = 0,$$

and denote its two roots by

$$\lambda_{1,2}(p,w)=\frac{1}{2(g_p)^2}\Big\{\big(-2g_w-g_p p\chi(w)\big)$$
$$\pm\Big[\big(2g_w+g_p\chi(w)\big)^2-\big(p\chi(w)\big)^2g_p^2\Big]^{1/2}\Big\}. \tag{5.6}$$

Since $h(p_*,w_*)=0$ and $h_p(p_*,w_*)>0$, we can find two stationary solutions $(\bar p_i,\bar w_i)$ ($\bar p_i$, $\bar w_i$ are constants) such that

$$\bar p_1<p_*<\bar p_2,\qquad \bar w_1<w_*<\bar w_2,$$
$$\frac{\bar p_1}{f(\bar w_1)}<q_0(x)<\frac{\bar p_2}{f(\bar w_2)},\qquad \bar w_1<w_0(x)<\bar w_2,$$

and

$$|\bar p_1-\bar p_2|\leqslant C\varepsilon,\qquad |\bar w_1-\bar w_2|\leqslant C\varepsilon.$$

By Theorem 2.1 we then have

$$\big|p(x,t)-p_*\big|\leqslant C\varepsilon,\qquad \big|w(x,t)-w_*\big|\leqslant C\varepsilon,$$

with another constant C and thus, in (5.6),

$$g_p=(\varphi h_p)(p_*,w_*)+O(\varepsilon),\qquad g_w=\varphi h_w(p_*,w_*)+O(\varepsilon).$$

Recalling (2.10), (2.11), we then easily see that the expression in brackets in (5.6) is positive for all ε sufficiently small and that both roots are positive. Hence (5.4) is satisfied by choosing $\lambda=(1/2)(\lambda_1(p_*,w_*)+\lambda_2(p_*,w_*))$, so that (5.5) holds.

In order to complete the proof of (2.32) we shall need the following lemma:

Lemma 5.1. *Let $k(t)$ be a function satisfying*

$$k(t)\geqslant 0,\qquad \int_0^\infty k(t)\,dt<\infty.$$

If either

(i) $|k'(t)|\geqslant C$, *or*
(ii) $|k(t+s)-k(t)|\leqslant\varepsilon(t)$, *for all* $s>0$, *where* $\varepsilon(t)\to 0$ *as* $t\to\infty$,

then $k(t)\to 0$, *as* $t\to\infty$.

Proof. If the assertion is not true then there is a sequence $t_n\to\infty$ such that

$$k(t_n)\to A,\qquad A>0.$$

In case (i) we get

$$k(t_n+s)>\frac{A}{2}\quad\text{if } |s|<\frac{A}{C},$$

and in case (ii) we get

$$\left|k(t_n+s)-k(t_n)\right|<\varepsilon(t)<\frac{A}{2},$$

if $n \geqslant n_0$ so that

$$k(t_{n_0}+s)>\frac{A}{2} \quad \text{if } s>0.$$

Thus, in both cases

$$\int_0^\infty k(t)\,dt=\infty,$$

which is a contradiction. □

Consider the function

$$k(t)=\int_\Omega \left(p(x,t)-\bar{p}\right)^2 dx.$$

By Poincare's inequality and (5.5)

$$\int_0^\infty k(t)\,dt \leqslant \int_0^\infty\int_\Omega \left|\nabla p(x,t)\right|^2 dx\,dt.$$

By (5.1) with $0, T$ replaced by $t, t+s$,

$$\left|\int_\Omega \left[p^2(x,t+s)-p^2(x,t)\right]dx\right|$$
$$\leqslant C\int_t^{t+s}\int_\Omega \left(\left|\nabla p(x,t')\right|^2+\left|\nabla w(x,t')\right|^2\right)dx\,dt'.$$

Using the relation

$$\int_\Omega \left[(p-\bar{p})^2(x,t+s)-(p-\bar{p})^2(x,t)\right]dx$$
$$=\int_\Omega \left[p^2(x,t+s)-p^2(x,t)\right]dx$$

and (5.5), we then have

$$\left|k(t+s)-k(t)\right| \leqslant \varepsilon(t) \quad \text{for all } s>0,$$

where $\varepsilon(t)\to 0$ if $t\to\infty$. Applying Lemma 5.1(ii), we obtain the first assertion in (2.32).

Similarly, setting

$$W(t) = \frac{1}{|\Omega|} \int_{\Omega} w(x,t)\,dx,$$

and introducing the function

$$k(t) = \int_{\Omega} \big(w(x,t) - W(t)\big)^2\,dx,$$

we deduce, by Poincare's inequality and (5.5), that

$$\int_0^\infty k(t)\,dt < \infty.$$

By the boundedness of w_t, $|k'| \leqslant C$. Hence, we may invoke Lemma 5.1(i) to conclude that

$$\int_{\Omega} \big|w(x,t) - W(t)\big|^2\,dx \to 0 \quad \text{if } t \to \infty. \tag{5.7}$$

Integrating (1.2) over Ω we get

$$W_t = \frac{1}{|\Omega|} \int_{\Omega} g(p,w)\,dx$$

so that

$$W_t - g(\bar{p}, W) = \varepsilon(t), \tag{5.8}$$

where

$$\big|\varepsilon(t)\big| \leqslant C\left\{\int_{\Omega} \big[(p-\bar{p})^2 + (w-W)^2\big]\,dx\right\}^{1/2}$$

and, as proved above,

$$\int_0^\infty \varepsilon^2(t)\,dt < \infty, \qquad \varepsilon(t) \to 0 \quad \text{as } t \to \infty. \tag{5.9}$$

Recalling (2.31) and introducing the function $\xi(t) = W(t) - \bar{w}$, we can rewrite (5.8) in the form

$$\frac{d\xi}{dt} + \big[g(\bar{p}, W) - g(\bar{p}, \bar{w})\big] = \varepsilon(t)$$

and, since by (2.10), (2.11), $g_w \sim \varphi h_w < 0$ for (p,w) near (p_*, w_*), we have

$$\frac{d\xi}{dt} + \sigma(t)\xi = \varepsilon(t),$$

where $\sigma \geqslant c > 0$. Using (5.9) we easily conclude that $\xi(t) \to 0$ if $t \to \infty$, so that

$$W(t) \to \bar{w} \quad \text{as } t \to \infty.$$

Combining this with (5.7), the second assertion in (2.32) follows. $\square$

The above proof, in case $\varphi \equiv 1$, clearly yields the assertion of Theorem 2.4.

Remark 5.1. By the Sobolev imbedding, for any $\delta > 0$ there holds

$$\left(\int_\Omega \left|w(x,t) - W(t)\right|^p dx\right)^{1/p} \leqslant \delta\left(\int_\Omega \left|\nabla w(x,t)\right|^2 dx\right)^{1/2} + C_\delta\left[\int_\Omega \left(w(x,t) - W(t)\right)^2 dx\right]^{1/2} \equiv I_\delta$$

if $1/p > 1/2 - 1/n$ if $n \geqslant 2$, and

$$\sup_{x\in\Omega}\left|w(x,t) - W(t)\right| \leqslant I_\delta \quad \text{if } n = 1.$$

Recalling that $\int_\Omega |\nabla w(x,t)|^2 dx < C$ for all $t > 0$, by (5.5), we easily conclude that

$$\left\{\int_\Omega \left|w(x,t) - \bar{w}\right|^p dx\right\}^{1/p} \to 0 \quad \text{if } t \to \infty, \text{ for } n \geqslant 2,$$

$$\sup_{x\in\Omega}\left|w(x,t) - \bar{w}\right| \to 0 \quad \text{if } t \to \infty, \text{ for } n = 1. \tag{5.10}$$

Remark 5.2. The proof (given in Section 2) that every stationary solution (p_*, w_*) is constant is valid, more generally, whenever

$$\frac{\partial h}{\partial p} \neq 0, \qquad p\chi(w)\frac{\partial h}{\partial p} + \frac{\partial h}{\partial w} \neq 0$$

holds for all $p_1 \leqslant p \leqslant p_2$, $w_1 \leqslant w \leqslant w_2$. However, the proofs of Theorems 2.1–2.5 require the inequality (2.11), and the inequality $\partial h/\partial p > 0$ if $\chi > 0$ or $\partial h/\partial p < 0$ if $\chi < 0$.

6. Extensions

We first state an extension of Theorems 2.1, 2.3 and 2.4 in case

$$\frac{\partial h}{\partial p} \geqslant 0, \quad p\chi\frac{\partial h}{\partial p} + \frac{\partial h}{\partial w} \leqslant 0 \quad \text{for } 0 < p_1 \leqslant p \leqslant p_2,\ w_1 \leqslant w \leqslant w_2 \tag{6.1}$$

provided $q_1 < q_2$.

Theorem 6.1. (i) *Theorems* 2.1 *and* 2.4 *remain valid if the strict inequalities* (2.10) *and* (2.11) *are replaced by* (6.1).

(ii) *If, in addition,*

$$(h_p)^2 + (p\chi h_p + h_w)^2 > 0, \tag{6.2}$$

then Theorem 2.4 *also holds.*

Indeed, the proof of Theorem 2.1 requires the strict inequalities in (2.23), but not strict inequalities in (2.10), (2.11). The proof of Theorem 2.2 is also unchanged. As for the proof of Theorem 2.4, the only point that needs to be observed is that as a consequence of (6.1), (6.2) we have $h_w < 0$ and, therefore, for any $\bar{p} \in (p_1, p_2)$ there is a unique $\bar{w}$ such that $h(\bar{p}, \bar{w}) = 0$.

The results of this paper extend to chemotaxis equations with several chemical species. Consider, for example, the system

$$\frac{\partial p}{\partial t} = \operatorname{div}\big(\nabla p - p\big(\chi_1(w_1)\nabla w_1 + \chi_2(w_2)\nabla w_2\big)\big), \quad x \in \Omega,\ t > 0, \tag{6.3}$$

$$\frac{\partial w_i}{\partial t} = \varphi(p, w_1, w_2) h_i(p, w_i), \quad x \in \Omega,\ t > 0\ (i = 1, 2), \tag{6.4}$$

with boundary conditions

$$\frac{\partial p}{\partial n} - p\left(\chi_1(w_1)\frac{\partial w_1}{\partial n} + \chi_2(w_2)\frac{\partial w_2}{\partial n}\right) = 0, \quad x \in \partial\Omega,\ t > 0, \tag{6.5}$$

where $\varphi(p, w_1, w_2) > 0$, $\chi_1(w_1) > 0$, $\chi_2(w_2) > 0$. Suppose

$$h_i(p_1, w_{i1}) = 0, \qquad h_i(p_2, w_{i2}) = 0,$$

where

$$0 \leqslant p_1 < p_2, \qquad w_{i1} < w_{i2} \quad (i = 1, 2),$$

and

$$\frac{\partial h_i}{\partial p} > 0, \tag{6.6}$$

$$\chi_1' g_1 + \chi_1 p \chi_1 g_{1,p} + \chi_1 g_{1,w_1} + \chi_2 p \chi_1 g_{2,p} < 0,$$

$$\chi_2' g_2 + \chi_2 p \chi_2 g_{2,p} + \chi_2 g_{2,w_2} + \chi_1 p \chi_2 g_{1,p} < 0 \tag{6.7}$$

for

$$p_1 \leqslant p \leqslant p_2, \qquad w_{i1} \leqslant w_i \leqslant w_{i2} \quad (i = 1, 2).$$

We introduce a variable q by

$$p = f_1(w_1) f_2(w_2) q, \tag{6.8}$$

where $f_i(w_i) = \exp[\int_{w_{i1}}^{w_i} \chi_i(s)\, ds]$. Setting

$$q_k = \frac{p_k}{f_1(w_{1k}) f_2(w_{2k})}, \tag{6.9}$$

we further assume that

$$q_1 < q_0(x) < q_2, \qquad w_{i1} < w_i(x,0) < w_{i2}. \tag{6.10}$$

Theorem 6.2. *Under the foregoing assumptions, there exists a unique solution* (p, w_1, w_2), *for all* $t > 0$, *and it satisfies the inequalities*

$$q_1 \leqslant q(x,t) \leqslant q_2, \qquad w_{i1} \leqslant w_i(x,t) \leqslant w_{i2} \quad (i = 1, 2).$$

The proof is similar to the proof of Theorems 2.1 and 2.2.

We finally extend some of the previous results to systems of the form (1.3). For simplicity we consider just the case where the system consists of (2.1) and

$$\frac{\partial w}{\partial t} = \varepsilon \Delta w + g(p, w), \quad x \in \Omega,\ t > 0, \tag{6.11}$$

where $0 < \varepsilon < 1$, with the boundary conditions (2.3) and

$$\frac{\partial w}{\partial n} = 0, \quad x \in \partial\Omega,\ t > 0, \tag{6.12}$$

and with the initial conditions (2.4). We assume that $\chi(w) > 0$. In order to extend Theorem 2.1 we introduce a function q by $p = f(w)q$, where

$$f(w) = \exp\left[\int_0^w \frac{1}{1-\varepsilon}\chi(s)\,ds\right]. \tag{6.13}$$

We also take $p_1 = w_1 = 0$ and define q_i as before, but replace (2.10), (2.11) by

$$h_p \geqslant 0 \quad \text{for } 0 \leqslant p \leqslant p_2,\ 0 \leqslant w \leqslant w_2, \tag{6.14}$$

$$p\chi h_p + (1-\varepsilon)h_w \leqslant 0 \quad \text{for } 0 \leqslant p \leqslant p_2,\ 0 \leqslant w \leqslant w_2. \tag{6.15}$$

We finally assume that

$$\chi^2 + (1-\varepsilon)\chi' \leqslant 0 \quad \text{if } 0 \leqslant w \leqslant w_2, \tag{6.16}$$

which implies, in particular, that $\chi'(w) < 0$.

Theorem 6.3. *Consider the system* (2.1), (6.11), (2.3), (6.12), (2.4). *Assume that, with* $f(w)$ *as in* (6.13), *the assumptions of Theorem* 2.1 *hold with* $p_1 = w_1 = 0$ *and with* (2.10), (2.11) *replaced by* (6.14)–(6.16). *Then there exists a unique global solution* (p, w) *such that*

$$0 < p \leqslant p_2, \qquad 0 \leqslant w \leqslant w_2, \tag{6.17}$$

and (2.25) *holds.*

Proof. We first extend Theorem 2.1. By the maximum principle we get $p > 0$ in Ω_T. Since $g(0,0) = 0$ and $g_p \geqslant 0$ we have $g(p,0) \geqslant 0$, so that, by the maximum principle, $w \geqslant 0$ in Ω_T. In order to prove (3.1) we compute the differential equation for q. Since

$$qf'w_t = qf'(\varepsilon\Delta w + g)$$

and

$$\varepsilon qf'\Delta w = \varepsilon\big[\operatorname{div}(qf'\nabla w) - f'\nabla q\cdot\nabla w - qf''|\nabla w|^2\big],$$

we get

$$\begin{aligned}\mathcal{L}q &\equiv q_t - \left[\Delta q + (1+\varepsilon)\frac{f'}{f}\nabla q\cdot\nabla w\right]\\ &= -qf'g\big(qf(w), w\big) + \varepsilon qf''|\nabla w|^2.\end{aligned}$$

From (6.13) and (6.16) we have

$$f'' = \frac{f}{(1-\varepsilon)^2}\big[\chi^2 + (1-\varepsilon)\chi'\big] \leqslant 0$$

so that

$$\mathcal{L}q \leqslant -qf'\varphi\big(qf(w), w\big)h\big(qf(w), w\big).$$

We can now proceed to prove (3.1) exactly as in the case of Theorem 2.1, making use of the inequalities (6.14), (6.15).

Having proved the inequalities (6.17) we next proceed to prove global existence and regularity of the solution, as in Theorem 2.2. In fact, the present situation is much simpler since w satisfies the heat equation with bounded source g. The proof that p and w are in $C^{\alpha,\alpha/2}_{x,t}(\Omega_T)$ can be obtained as before, and the $C^{\alpha,\alpha/2}_{x,t}(\Omega_T)$ bound is now independent of T, $0 < T < \infty$. We can then use the Schauder estimates, first for w and then for p, in order to complete the proof. □

Remark 6.1. The Schauder estimates imply, in particular, that

$$|\nabla w| \leqslant C, \qquad |\nabla p| \leqslant C, \qquad |w_t| \leqslant C, \qquad |p_t| \leqslant C$$

uniformly in Ω_∞.

We next state an extension of Theorem 2.3 assuming, for simplicity, that

$$g(p, w) = p - \mu w \quad (\mu > 0). \tag{6.18}$$

In this case (6.14) holds, and (6.15) reduces to the inequality $p\chi \leqslant (1-\varepsilon)\mu$, which certainly holds if $\chi(0)p_2 \leqslant (1-\varepsilon)\mu$. We need to make one additional assumption.

Let us denote by $C(\Omega)$ the smallest positive constant such that

$$\int\limits_\Omega u^2\,dx \leqslant C(\Omega)\int\limits_\Omega |\nabla u|^2\,dx$$

for all functions $u(x)$ in $H^1(\Omega)$ with $\int_\Omega u(x)\,dx = 0$. Note that if $\Omega = \rho\Omega_0$ (Ω_0 a fixed domain and $0 < \rho < 1$) then $C(\Omega) = C(\Omega_0)\rho^2$.

The additional condition we need is

$$C(\Omega)\big(\chi(0)p_2\big)^2 < 4\mu\varepsilon. \tag{6.19}$$

Theorem 6.4. *Let the assumptions of Theorem* 6.3 *and the additional assumptions* (6.18), (6.19) *hold. Then the solution* (p, w) *satisfies the asymptotic behavior* (2.32) *with* $w = \bar{p}/\mu$.

Proof. Set

$$W(t) = \frac{1}{|\Omega|}\int_\Omega w(x,t)\,dx.$$

Integrating (6.11) over Ω we get $W' + \mu W = \bar{p}$, so that

$$W(t) = \frac{\bar{p}}{\mu} + c_0 e^{-\mu t} \quad \left(c_0 = W(0) - \frac{\bar{p}}{\mu}\right). \tag{6.20}$$

We can write

$$\begin{aligned}
&\int_\Omega (p - \mu w)w\,dx \\
&\quad = \int_\Omega \big[(p - \mu w) - (\bar{p} - \mu W)\big](w - W)\,dx - \int_\Omega W(\bar{p} - \mu W)\,dx \\
&\quad = \int_\Omega (p - \bar{p})(w - W)\,dx - \int_\Omega \mu(w - W)^2\,dx + O(e^{-\mu t})
\end{aligned}$$

by (6.20). By Schwarz's inequality the first integral on the right-hand side is bounded by

$$\int_\Omega \mu(w - W)^2\,dx + \frac{1}{4\mu}\int_\Omega (p - \bar{p})^2\,dx.$$

Hence

$$\int_\Omega (p - \mu w)w\,dx \leqslant \frac{1}{4\mu}\int_\Omega (p - \bar{p})^2\,dx + O(e^{-\mu t}),$$

and by Poincare's inequality we then get

$$\int_\Omega (p - \mu w)w\,dx \leqslant C(\Omega)\frac{1}{4\mu}\int_\Omega |\nabla p|^2\,dx + O(e^{-\mu t}). \tag{6.21}$$

We now proceed with (5.1) as before. Similarly

$$\frac{1}{2}\int_{\Omega} w^2\Big|_0^T\,dx + \varepsilon\iint_{\Omega_T} |\nabla w|^2\,dx\,dt = \iint_{\Omega_T} g(p,w)w\,dx\,dt. \tag{6.22}$$

Multiplying the expression (5.1) by a positive constant λ and adding to (6.22), we obtain

$$\begin{aligned}
&\frac{\lambda}{2}\int_{\Omega} p^2\Big|_0^T\,dx + \lambda\iint_{\Omega_T} |\nabla p|^2\,dx\,dt + \frac{1}{2}\int_{\Omega} w^2\Big|_0^T\,dx + \varepsilon\iint_{\Omega_T} |\nabla w|^2\,dx\,dt \\
&\quad = \lambda\iint_{\Omega_T} p\chi(w)\nabla p\cdot\nabla w\,dx\,dt + \iint_{\Omega_T} g(p,w)w\,dx\,dt
\end{aligned} \tag{6.23}$$

and $p\chi \leqslant p_2\chi(0) \equiv A$. By Schwarz's inequality

$$\begin{aligned}
&\lambda\iint_{\Omega_T} p\chi(w)\nabla p\cdot\nabla w\,dx\,dt \\
&\quad \leqslant \varepsilon\iint_{\Omega_T} |\nabla w|^2\,dx\,dt + \lambda^2\frac{A^2}{4\varepsilon}\iint_{\Omega_T} |\nabla p|^2\,dx\,dt.
\end{aligned} \tag{6.24}$$

Substituting (6.24) and (6.21) into (6.23) we get

$$\left(\lambda - \lambda^2\left(\frac{A^2}{4\varepsilon}\right) - \frac{C(\Omega)}{4\mu}\right)\iint_{\Omega_T} |\nabla p|^2\,dx\,dt \leqslant C. \tag{6.25}$$

Due to the inequality (6.19), the quadratic equation

$$\frac{A^2}{4\varepsilon}\lambda^2 - \lambda + \frac{C(\Omega)}{4\mu} = 0$$

has two positive roots, say $0 < \lambda_1 < \lambda_2$. Choosing any $\lambda \in (\lambda_1, \lambda_2)$ we obtain from (6.25) the inequality

$$\iint_{\Omega_T} |\nabla p|^2\,dx\,dt \leqslant C.$$

By (6.22) and (6.21) we then also get the same bound in ∇w, so that

$$\iint_{\Omega_\infty} |\nabla p|^2\,dx\,dt + \iint_{\Omega_\infty} |\nabla w|^2\,dx\,dt \leqslant C.$$

We can now proceed as in the proof of Theorem 2.3, noting, by Remark 6.1, that w_t is bounded. $\square$

Remark 6.2. From Remark 5.1 and (2.32) it follows that

$$p(x,t) - \bar{p} \to 0, \qquad w(x,t) - \bar{w} \to 0 \tag{6.26}$$

uniformly in Ω_∞ as $t \to \infty$.

Remark 6.3. Since any stationary solution (p_*, w_*) of (2.1), (6.11), (2.3), (6.12) with $0 < w_*(x) < w_2$, $0 < p_*(x) < p_2$ satisfies the estimate

$$\iint\limits_{\Omega_\infty} |\nabla p|^2 \, dx\, dt + \iint\limits_{\Omega_\infty} |\nabla w|^2 \, dx\, dt \leqslant C,$$

it follows that such solutions are necessarily constant.

Remark 6.4. Gajewsky and Zacharias [6] considered a system of two parabolic equations in a two-dimensional domain which, after some normalization, reduces to the system considered in Theorem 6.4 with $\chi(w) \equiv 1$. Making smallness assumptions analogous to (6.19) they constructed Lyapunov functions and proved that global solutions exist and satisfy the asymptotic behaviour (6.26).

7. Examples

Example 1.

$$w_t = p - \mu h(w) \quad (\mu > 0). \tag{7.1}$$

The condition (2.10) is satisfied. If

$$\chi(w)h(w) < h'(w) \quad \text{for } p_1 \leqslant p \leqslant p_2,\ w_1 \leqslant w \leqslant w_2 \tag{7.2}$$

and $w_2 - w_1$ is sufficiently small, then (2.11) is also satisfied and Theorems 2.1–2.3 can be applied. Thus, in particular, any stationary solution is stable.

If (7.2) holds, but $w_2 - w_1$ is not necessarily small, then the condition (2.11) can be stated in the form

$$\frac{h(w_2)}{f(w_2)} f(w)\chi(w) < h'(w) \quad \text{for all } w_1 \leqslant w \leqslant w_2. \tag{7.3}$$

Theorem 2.4 implies that if (7.2), (7.3) hold then any solution satisfying (2.23) has the asymptotic behavior (2.32). The special case

$$w_t = p - \mu w$$

was discussed in [19] where it was proved that if $n = 1$ and

$$\chi(w) = \frac{\beta}{\alpha + \beta w} \quad (\alpha > 0,\ \beta > 0)$$

then stationary solutions are linearly stable. Since in this case the condition (7.2) is satisfied, Theorem 2.3 implies that stationary solutions are in fact asymptotically stable.

Example 2.

$$g(p,w)=\left(\frac{p}{1+\nu w}-\mu\right)w+\gamma\frac{p}{p+1}, \tag{7.4}$$

$$\chi(p,w)=\frac{\alpha}{(w+\alpha)(w+\beta)}, \tag{7.5}$$

where the constants ν,μ,γ and α,β are positive.

In this example the production of the chemical species w is affected by the population concentration p. Such a model reflects conditions encountered in instances such as myxobacteria gliding on slime trails [19].

We can write

$$g(p,w)=\frac{w}{(1+\nu w)(p+1)}h(p,w),$$

where $h(p,w)$ is a quadratic polynomial in p with the unique positive root

$$p=\tilde{h}(w)\equiv\frac{1}{2}\left\{-\left(1-\mu(1+\nu w)+\frac{\gamma}{w}+\nu\gamma\right)\right.$$
$$\left.+\left[\left(1-\mu(1+\nu w)+\frac{\gamma}{w}+\nu\gamma\right)^2+4\mu(1+\nu w)\right]^{1/2}\right\}.$$

Assume that $w_1>4\gamma/\mu$. If γ is sufficiently large then for any $w_2>w_1$ there holds

$$p_2\equiv\tilde{h}(w_2)>\tilde{h}(w_1)\equiv p_1,$$
$$g(p,h)=\varphi(p,w)\big(p-\tilde{h}(w)\big)\quad\text{for } p_1\leqslant p\leqslant p_2,\ w_1\leqslant w\leqslant w_2,$$

where $\varphi(p,w)>0$ and the function $h(p,w)\equiv p-\tilde{h}(w)$ satisfies the condition (2.11). Hence stationary solutions (p_*,w_*) with $p_1<p_*<p_2$, $w_1<p_*<w_2$ are asymptotically stable. Linear stability for the case $\gamma=0$ was proved in [19].

Example 3.

$$w_t=(p-\mu)w,\qquad \chi(w)=\frac{1}{w}. \tag{7.6}$$

Taking $h(p,w)=(p-\mu)w$, $w_1=p_1=0$, $p_2=\mu$ and w_2 arbitrarily large, the inequality $h_p\geqslant 0$ holds, but $p\chi h_p+h_w=2p-\mu>0$ if $\mu/2<p\leqslant\mu$, so that the second condition in (6.1) is not satisfied. In this case spatially independent solutions are linearly unstable [19] and, moreover, there exist solutions that blow up in finite time [15]. However, for the problem with

$$w_t=(p-\mu)w,\qquad \chi(w)=-\frac{1}{w} \tag{7.7}$$

in dimension $n = 1$, global solutions exist for any $C^{2+\beta}$ initial data (see [5]). This problem, which arises in a model of initiation of angiogenesis [16], is not covered by Theorem 2.5.

Example 4.

$$w_t = p(\mu - w) \quad (0 < \mu \leqslant 1), \qquad \chi(w) = 1. \tag{7.8}$$

Take $h = p(\mu - w)$, $p_1 = w_1 = 0$, $w_2 = \mu$ and p_2 arbitrarily large. Although $h_p = h_w = 0$ at $p = 0$, $w = \mu$, since $\mathcal{L}q + (\chi h)q = 0$ (see (2.18)) and $\chi h \geqslant 0$,

$$q(x,t) \leqslant \max_{x\in\Omega}\{q_0(x)\} > 0$$

by the maximum principle. Consequently, the solution (p, w) avoids the point $(0, \mu)$ so that Theorem 6.1 can be used to conclude that if $0 < w_0(x) < \mu$ and $p_0(x) > 0$ then there exists a unique global solution satisfying (2.32) with $\bar{w} = \mu$.

Example 5.

$$w_t = -pw, \qquad \chi(w) = \frac{1}{w}. \tag{7.9}$$

This case is not included in Theorem 6.1. Nonetheless, we can use the transformation $p = qw$ (as before) to write, analogously to (2.18),

$$\mathcal{L}q = wq^2.$$

Then, if $0 < q_1 \leqslant q_0(x) \leqslant q_2$ and $0 < w_0(x) < \varepsilon$ we have, by comparison,

$$q(x,t) \geqslant q_1 \quad \text{and, therefore,} \quad w \leqslant \varepsilon e^{-q_1 t}.$$

Again by comparison, $q(x,t) \leqslant Q(t)$, where Q satisfies the system

$$Q' = \varepsilon e^{-q_1 t} Q^2, \qquad Q(0) = q_2,$$

whose solution is given by

$$\frac{1}{Q} = \frac{1}{q_2} - \frac{\varepsilon}{q_1}(1 - e^{-q_1 t}).$$

If $\varepsilon < q_1/q_2$ then $Q(t)$ remains bounded and there exists a global solution (p, w) with pw uniformly bounded.

Example 6. We consider a model of tumor induced angiogenesis of Anderson and Chaplain [3]. Let p denote the density of the endothelial cells, c the concentration of the tumor angiogenesis factor (secreted by the tumor) and w the density of the fibronectin cells. Then

$$\frac{\partial p}{\partial t} = \operatorname{div}\left(\nabla p - p\left(\frac{\alpha}{1+c}\nabla c + \rho \nabla w\right)\right), \tag{7.10}$$

$$\frac{\partial w}{\partial t} = \gamma p(1-w), \tag{7.11}$$
$$\frac{\partial c}{\partial t} = -\mu p c, \tag{7.12}$$

where α, ρ, γ and μ are positives constants. We assume that

$$c_0(x) > 0, \qquad 0 < w_0(x) < 1, \qquad q(x,0) \geqslant q_1 > 0.$$

This system does not satisfy the conditions of Theorem 6.2. However, by performing the transformation (6.8), we can prove that if

$$\frac{\mu}{\gamma} \leqslant 1 \quad \text{and} \quad \bigl(1 - w_0(x)\bigr)^{\mu/\gamma} < \delta c_0(x)$$

for some positive constant δ which depends only on α, ρ, γ, μ then $\mathcal{L}q \geqslant 0$, where $\mathcal{L}$ is a parabolic operator with no zero order terms, so that $q(x,t) \geqslant q_1$. If we further assume that $c_0(x) < \varepsilon$ where ε is sufficiently small then we can use comparison, as in Example 5, to conclude that there exists a unique global bounded solution.

We also note that in the special one-dimensional case with

$$c_0(x) = k(1-x) \quad (k \text{ constant}), \qquad w_0(x) = 1 - (1-x)^{\mu/\gamma}$$

we have $c = k(1-w)^{\gamma/\mu}$. Therefore p, w satisfy the system (2.1)–(2.4) with

$$\chi(w) = \rho - \frac{\alpha}{1 + k(1-w)^{\mu/\gamma}} \frac{\mu}{\gamma} k(1-w)^{\mu/\gamma - 1}. \tag{7.13}$$

Taking $h = p(1-w)$, $w_1 = p_1 = 0$, $w_2 = 1$ and p_2 arbitrarily large, one can verify that if

$$\frac{\mu}{\gamma} \geqslant 1 \quad \text{and} \quad \frac{\mu}{\gamma} \alpha k < \rho < 1, \tag{7.14}$$

then (by Theorems 2.1, 2.2, 2.4) there exists a unique global solution having the asymptotic behavior (2.32). (The fact that $h_p = h_w = 0$ at $(0, \mu)$ does not cause any difficulties; cf. Example 4.)

Acknowledgments

The first author is partially supported by the National Science Foundation Grant DMS 9970522. The second author is partially supported by projects DGES (Spain) REN 2000/0766 and HPRN-CT-2000-00105.

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